New results concerning the stability of equilibria of a delay differential equation modeling leukemia

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Abstract

The paper is devoted to the study of stability of equilibria of a delay differential equation that models leukemia. The equation was previously studied in [5] and [6], where the emphasis is put on the numerical study of periodic solutions. Some stability results for the equilibria are also presented in these works, but they are incomplete and contain some errors. Our work aims to complete and to bring corrections to those results. Both Lyapunov first order approximation method and second Lyapunov method are used.
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1 Introduction

The study of the mathematical model of periodic chronic myelogenous leukemia considered in [5], [6] may be reduced to that of the equation

$$\dot{x}(t) = -\left[\frac{\beta_0}{1+x(t)^n} + \delta\right] x(t) + k\frac{\beta_0 x(t-r)}{1+x(t-r)^n}, \tag{1}$$

where β_0, n, δ, k, r are positive parameters and $k = 2e^{-\gamma r}$, with $\gamma > 0$. We do not insist here on the significance of the function $x(.)$ or in that of the parameters, since these are extensively presented in [5], [6]. We only remind that the unknown function, $x(\cdot)$, should be nonnegative, being a non-dimensional density of cells.

The two studies cited above are mainly devoted to the numerical investigation of the delay differential equation above. The stability of equilibria study in [5], [6], that is reduced to a few lines, is, unhappily, incomplete and contains some errors.

Our work aims to correct the errors in the stability conditions presented in [5], [6] and to present some aspects concerning the dynamics generated by equation (1), aspects that were not pointed out there.

In Subsection 1.1 we prove that the Cauchy problem associated to our equation has an unique defined on $[-r, \infty)$ bounded solution. In Subsection 1.2, following [5], [6], the two equilibrium solutions, as well as the linearized equation and the characteristic equation for each of these, are presented. Section 2 deals with the stability study of the equilibrium points. We use the results of [2] and, in a first stage, we obtain results valid for both equilibria. In the subsequent subsections we analyze the stability of the two points individually. It is important to perform this separate study since the conclusions are very specific to each equilibrium point. In Section 3 we comment the stability results in [5], [6], pointing out the errors therein.

1.1 Existence and uniqueness of solution

We make the notation $\mathcal{B} = C([-r, 0], \mathbb{R})$ (the space of continuous, real valued functions defined on $[-r, 0]$, with the supremum norm, denoted by $|x|_0$). Given a function $x : [-r, T) \mapsto \mathbb{R}$, $T > 0$ and a $0 \leq t < T$, we define the function $x_t \in \mathcal{B}$ by $x_t(s) = x(t + s)$.

Equation (1) may be written as

$$\dot{x} = h(x_t), \tag{2}$$

where $h : \mathcal{B} \mapsto \mathbb{R}$, and we impose to this equation the initial condition

$$x_0 = \phi \in \mathcal{B}. \tag{3}$$

Remark that if the initial condition is a positive function, then $x(t)$ can not become strictly negative. Indeed, let t_1 be the first moment when $x(t_1) = 0$, (that is $x(t) > 0$ for $t < t_1$). Then $\dot{x}(t_1) = k\frac{\beta_0 x(t_1-r)}{1+x(t_1-r)^n} > 0$, hence $x(t) > 0$ for $t \geq t_1$ in a neighborhood of t_1.

We study the existence, uniqueness and domain of existence of solutions.

The function h is globally Lipschitz. Indeed, by denoting $\beta(x) = \beta_0/(1 + x^n)$, we have

$$|\frac{d}{dx}(\beta(x)x)| < \beta_0(n + 1),$$

and thus, for any φ_1, $\varphi_2 \in \mathcal{B}$

$$|h(\varphi_1)-h(\varphi_2)| \leq (\beta_0(n+1)+\delta)|\varphi_1(0)-\varphi_2(0)|+k\beta_0(n+1)|\varphi_1(-r)-\varphi_2(-r)| \leq$$

$$\leq [(k + 1)\beta_0(n + 1) + \delta]|\varphi_1 - \varphi_2|_0.$$

From here the continuity of h follows also.

Theorem 2.3 of [3] implies that problem (1), (3) has an unique solution defined on an interval $[0, T)$.

If we take $\varphi_2 = 0$, by using the fact that $h(0) = 0$, we obtain that

$$|h(\varphi_1)| \leq [(k+1)\beta_0(n+1) + \delta]|\varphi_1|_0$$

thus the function h is also completely continuous.

We prove that the solution is bounded. For this, we multiply equation (1) by $x(t)$ and we get

$$\dot{x}(t)x(t) = -\beta_0 \frac{x^2(t)}{1+x^n(t)} - \delta x^2(t) + k\beta_0 \frac{x(t-r)x(t)}{1+x^n(t-r)} \leq$$

$$\leq -\delta x^2(t) + \frac{\varepsilon}{2} k\beta_0 \frac{x^2(t)}{1+x^n(t-r)} + \frac{1}{2\varepsilon} k\beta_0 \frac{x^2(t-r)}{1+x^n(t-r)}$$

$$\leq -\delta x^2(t) + \frac{\varepsilon}{2} k\beta_0 x^2(t) + \frac{1}{2\varepsilon} k\beta_0,$$

and

$$\frac{d(x^2(t))}{dt} + (2\delta - \varepsilon k\beta_0)x^2(t) \leq \frac{k\beta_0}{\varepsilon}.$$

We chose an $\varepsilon > 0$ such that $\eta := 2\delta - \varepsilon k\beta_0 > 0$ and we obtain by integration

$$x^2(t) \leq \phi^2(0)e^{-\eta t} + \frac{k\beta_0}{\varepsilon\eta},$$

hence the solution of problem (1), (3) is bounded. Theorem 3.2 of [3] implies that the solution is defined on the whole positive real time semiaxis.

Hence, for any $\phi \in \mathcal{B}$, problem (1), (3) has an unique defined on $\mathbb{R}^+$ bounded solution. We can thus associate to this problem the semigroup of operators on $\mathcal{B}$, $\{T(t)\}_{t\geq 0}$, $T(t)(\phi) = x_t(\phi)$, where $x(t, \phi)$ is the solution of eq. (1) with initial condition $x_0 = \phi$.

1.2 Equilibrium solutions

In this subsection we, inevitably, follow [5].

The equilibrium points of the problem are

$$x_1 = 0,\ x_2 = (\frac{\beta_0}{\delta}(k-1) - 1)^{1/n}.$$

The second one is acceptable from the biological point of view if and only if it is strictly positive that is, if and only if

$$\frac{\beta_0}{\delta}(k-1) - 1 > 0. \tag{4}$$

In terms of r, by using $k = 2e^{-\gamma r}$, the above inequality may be written as

$$r < r_{max} := -\frac{1}{\gamma} \ln \frac{1}{2} \left(1 + \frac{\delta}{\beta_0}\right), \tag{5}$$

and since the delay r is positive, the condition $\delta/\beta_0 < 1$ follows.

The biological interpretation of function β [5] shows that the condition $\beta(x_2) = \delta/(k-1) > 0$ should be fulfilled. This is equivalent to $k > 1$.

The linearized equation around one of the equilibrium points is

$$\dot{z}(t) = -[B + \delta]z(t) + kBz(t - r), \tag{6}$$

with $B = \beta'(x^*)x^* + \beta(x^*)$, $x^* = x_1$ or $x^* = x_2$.

The eigenvalues of the infinitesimal generator of the semigroup of operators generated by equation (6) are the solutions of the characteristic equation

$$\lambda + \delta + B = kBe^{-\lambda r}. \tag{7}$$

2 Stability of equilibrium points

In order to investigate the stability of the equilibrium solutions, we first try to establish the conditions in which all the eigenvalues have strictly negative real part, in order to use the linear approximation Lyapunov method.

We rely on the work [2] that exhaustively solves the problem of finding necessary and sufficient conditions on the parameters such that the equation $\lambda = a_1 + a_2 e^{-\lambda}$ has only solutions with strictly negative real part.

We denote $\delta + B = p$, $kB = q$, hence (7) becomes

$$\lambda + p = qe^{-\lambda r}. \tag{8}$$

By taking $\lambda = \mu + i\omega$, and by equating the real, resp. the imaginary parts in our equation, we obtain

$$\mu + p = qe^{-\mu r}\cos(\omega r), \tag{9}$$

$$\omega = -qe^{-\mu r}\sin(\omega r).$$

It is useful to consider the case $\mu = 0$ in the above equations,

$$p = q\cos(\omega r), \tag{10}$$

$$\omega = -q\sin(\omega r).$$

The results in [2] imply the following

Proposition *All solutions λ of eq. (8) satisfy $Re\lambda < 0$, if and only if*
a) $q < 0$, $0 < -pr < 1$ *and* $-p < -q < (\omega_0^2 + p^2)^{1/2}$,
or

b) *$q < 0$, $p > 0$ and $-q < (\omega_0^2 + p^2)^{1/2}$,*
or
c) *$q > 0$, $p > 0$, and $q < p$,*
where ω_0 is the solution in $(0, \pi/r)$ of the equation

$$\omega \cot(\omega r) = -p. \tag{11}$$

Remark. If we divide the first equality in (10) to the second one, we obtain (11). Hence relations (10) are equivalent to the set of relations (11) and $\omega^2 + p^2 = q^2$.

In order to express ω_0 in a more direct form, we consider the function $T : [0, \pi) \mapsto (-\infty, 1]$, given by

$$T(y) = \begin{cases} y \cot(y), & y \in (0, \pi); \\ 1, & y = 0. \end{cases} \tag{12}$$

The function is a bijection and we can equivalently define ω_0, the solution of (11), as

$$\omega_0 = \frac{1}{r} T^{-1}(-pr). \tag{13}$$

We express the conditions in Proposition in terms of r. We first remark that

$$\omega_0^2 + p^2 = \frac{p^2}{\cot^2(\omega_0 r)} + p^2 = \frac{p^2}{\cos^2(\omega_0 r)}.$$

The two inequalities in ***a)*** of the above Proposition may be written as

$$|p| < |q| < \frac{|p|}{|\cos(\omega_0 r)|}.$$

Since $p < 0$, the solution ω_0 of equation (11) is such that $\omega_0 r \in (0, \pi/2)$. Hence the above inequality is equivalent to

$$0 < \frac{p}{q} < 1, \, \cos(\omega_0 r) < \frac{p}{q},$$

and the second one is equivalent to $\arccos(\frac{p}{q}) < \omega_0 r < \pi/2$. To conclude, case **a)** is described by the inequalities

$$q < p < 0, \frac{\arccos(\frac{p}{q})}{\omega_0} < r < \frac{1}{|p|}. \tag{14}$$

In case **b)** $q < 0$, $p > 0$, and we must have

$$-q < \frac{p}{|\cos(\omega_0 r)|}.$$

In this case, $\omega_0 \cot \omega_0 r = -p < 0$, and since $\omega_0 r \in (0, \pi)$, we must have $\omega_0 r \in (\pi/2, \pi)$. The above inequality is equivalent to

$$|\cos(\omega_0 r)| < \frac{p}{|q|}$$

and this one is satisfied if

$$p/|q| > 1 \text{ or } \{p/|q| \leq 1 \text{ and} \, (-\cos(\omega_0 r) < \frac{p}{-q})\}.$$

The last condition is equivalent to

$$-1 \leq \frac{p}{q} < 0 \, \text{and} \, \cos(\omega_0 r) > \frac{p}{q} \Leftrightarrow \frac{\pi}{2} < \omega_0 r < \arccos(\frac{p}{q}) \Leftrightarrow$$

$$\Leftrightarrow \frac{\pi}{2\omega_0} < r < \frac{\arccos(\frac{p}{q})}{\omega_0}.$$

Remark. The case $q < 0$, $p = 0$ corresponds to $\omega_0 r = \pi/2$, and the eigenvalues lie to the left of the vertical axis if and only if $-qr < \pi/2$.

We can now translate the discussion above to our concrete problem.

I. If $B < 0$, then two situations may occur.

A. $\delta + B < 0$. In this situation, $Re\lambda < 0$ for all eigenvalues λ if and only if $|\delta + B| < |kB|$ and

$$\frac{\arccos\left((\delta + B)/kB\right)}{\omega_0} < r < \frac{1}{|\delta + B|}, \tag{15}$$

where

$$\omega_0 = \frac{1}{r} T^{-1}(-(\delta + B)r),$$

with T given by (12).

If the studied equilibrium point is x_2, the condition $r \leq r_{max}$ must be also fulfilled.

B. $\delta + B > 0$. In this situation, $Re\lambda < 0$ for all eigenvalues λ if and only if

$$\delta + B > |kB| \text{ or } \left\{ \delta + B \leq |kB| \text{ and } r < \frac{\arccos\left((\delta + B)/kB\right)}{\omega_0} \right\} \tag{16}$$

where, again

$$\omega_0 = \frac{1}{r} T^{-1}(-(\delta + B)r),$$

with T given by (12).

II. If $B > 0$, then we can only have $\delta + B > 0$, and in this situation $Re\lambda < 0$ for all eigenvalues λ if and only if

$$kB < \delta + B.$$

Even if the above discussion seems comprehensive, it is still useful to consider the two equilibrium points separately and to discuss their stability.

2.1 Stability properties of x_1

In this case, $B = \beta_0 > 0$, hence the necessary and sufficient condition for the negativity of the real part of all eigenvalues is

$$k\beta_0 < \delta + \beta_0 \Leftrightarrow \frac{\beta_0}{\delta}(k-1) < 1.$$

Since the condition (4) for the existence of the second equilibrium point, x_2, is the reverse of the above inequality, it follows that x_1 is stable as long as it is the single equilibrium point. When the second equilibrium point occurs, x_1 becomes unstable.

We inspect the eigenvalues at $\frac{\beta_0}{\delta}(k-1) = 1$. Equation (7) in this case is

$$\lambda + \delta + \beta_0 = k\beta_0 e^{-\lambda r},$$

and, since $k\beta_0 = \delta + \beta_0$, admits the solution $\lambda = 0$. Hence the change of stability occurs by traversing the eigenvalue $\lambda = 0$.

2.1.1 Stability of x_1 when $\frac{\beta_0}{\delta}(k-1) = 1$

In this case, the "first order approximation" theorem is of no use, since 0 is the eigenvalue with greatest real part. We use a Lyapunov function in order to prove stability of the zero solution.

However, since for our problem $x(t) \geq 0$, the concept of stability should be interpreted in the following way:
for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\phi(s) \geq 0$, $s \in [-r, 0]$, and $|\phi|_0 < \delta$, then $0 \leq x(t,\phi) < \varepsilon$ for any $t > 0$, where, as above, $x(t,\phi)$ is the solution of (1) with condition (3).

If $V : \mathcal{B} \mapsto \mathbb{R}$ is continuous, the derivative along the solution $x(\cdot,\phi)$ of the Cauchy problem (2), (3) is defined as [3]

$$\dot{V}(\phi) = \limsup_{h\to 0^+} \frac{1}{h}[V(x_h(\phi)) - V(\phi)].$$

Definition [3]. *V is a Lyapunov function on $G \subset \mathcal{B}$ if V is continuous on $\overline{G}$ and $\dot{V} \leq 0$ on G.*

Theorem [1]. *If $V : \mathcal{B} \mapsto \mathbb{R}$ is a Lyapunov function and there exist a continuous increasing function $a : [0, \infty) \mapsto [0, \infty)$, with $a(0) = 0$ and*

$$a(|\phi(0)|) \leq V(\phi),$$

then the solution $x = 0$ of equation (2) is stable and every solution is bounded.

We construct below a Lyapunov function for our problem, for the considered parameter values.

Let us consider the function $G(u) = \int_0^u 2s/(1+s^n)ds$. We define

$$V(\phi) = G(\phi(0)) + k\beta_0 \int_{-r}^{0} \frac{\phi^2(s)}{(1+\phi^n(s))^2} ds.$$

We have

$$\dot{V}(\phi) = \frac{2\phi(0)}{1+\phi^n(0)} \dot{x}(0,\, \phi) + k\beta_0 \left[\frac{\phi^2(0)}{(1+\phi^n(0))^2} - \frac{\phi^2(-r)}{(1+\phi^n(-r))^2} \right],$$

and by using the equality

$$\dot{x}(0, \phi) = -[\beta(\phi(0)) + \delta]\phi(0) + k\beta(\phi(-r))\phi(-r),$$

we obtain

$$\dot{V}(\phi) = -2\beta_0 \frac{\phi^2(0)}{(1+\phi^n(0))^2} - 2\delta \frac{\phi^2(0)}{1+\phi^n(0)} + 2k\beta_0 \frac{\phi(0)\phi(-r)}{(1+\phi^n(0))(1+\phi^n(-r))} +$$

$$+k\beta_0 \left[\frac{\phi^2(0)}{(1+\phi^n(0))^2} - \frac{\phi^2(-r)}{(1+\phi^n(-r))^2} \right],$$

from where, with the inequality

$$\frac{2\phi(0)\phi(-r)}{(1+\phi^n(0))(1+\phi^n(-r))} \leq \frac{\phi^2(0)}{(1+\phi^n(0))^2} + \frac{\phi^2(-r)}{(1+\phi^n(-r))^2}$$

we obtain

$$\dot{V}(\phi) \leq 2(-\beta_0 - \delta + k\beta_0) \frac{\phi^2(0)}{(1+\phi^n(0))^2} = 0,$$

since $k\beta_0 = \delta + \beta_0$.

The hypotheses of Theorem 1 are satisfied with $a(u) = G(u)$, and it follows that $x_1 = 0$ is stable in the case of this subsection.

2.2 Stability properties of x_2

In this case,

$$B = \beta_0[n - (n-1)A]/A^2 \tag{17}$$

where $A = \beta_0(k-1)/\delta$.

As pointed out in Subsection 1.2, in this case the condition (5) must be fulfilled.

We refine the discussion concerning the cases of stability given at the beginning of Section 2, for this concrete B.

I.A. The condition $B < 0$ and the definition of B imply $n-(n-1)A < 0$. This implies $n > 1$ and

$$\frac{\beta_0}{\delta}(k-1) > \frac{n}{n-1}. \tag{18}$$

The condition $B+\delta<0$ leads to $n-(n-k)A<0$, that implies $n>k$ and

$$\frac{\beta_0}{\delta}(k-1)>\frac{n}{n-k}. \tag{19}$$

Obviously, the second inequality implies the first one.

The sufficient condition of local stability is condition (15), with B given by (17). We remark that the condition $|\delta+B|<|kB|$ is satisfied since it is equivalent to $\delta+B>kB$ and this one is equivalent to $\frac{\beta_0}{\delta}(k-1)>1$, (the condition of positivity of x_2).

We have to study the behavior of the solutions at the extremities of the interval of stability.

a) We consider the case

$$r=\frac{\arccos((\delta+B)/(kB))}{\omega_0}. \tag{20}$$

This relation, together with

$$\omega_0\cot(\omega_0 r)=-(\delta+B), \tag{21}$$

and $\omega_0>0$ (from the definition of ω_0), imply

$$\omega_0=\sqrt{(kB)^2-(\delta+B)^2} \tag{22}$$

and that the pair $\mu^*=0$, $\omega^*=\omega_0$ represents a solution of (9).

For later use we remark that, for $B<0$, the relations (21) and (22) (where, by the definition of ω_0, $\omega_0 r\in(0,\pi)$) together, imply relation (20) and again that the pair $\mu^*=0$, $\omega^*=\omega_0$ is a solution of (9).

We assume that we vary one of the parameters, that we denote here by α, such that for a value α^* the equality (20) is satisfied, and keep all other parameters fixed. We then obtain two complex conjugated branches of eigenvalues $\lambda_{1,2}(\alpha)=\mu(\alpha)\pm i\omega(\alpha)$, such that $\lambda_{1,2}(\alpha^*)=\pm i\omega^*$. If $\frac{d\mu}{d\alpha}(\alpha^*)\neq 0$ and the first Lyapunov coefficient of the reduced on the center manifold at α^* equation is different from zero, then a Hopf bifurcation takes place in the center manifold. The sign of the first Lyapunov coefficient gives the stability properties of the solution at α^* and of the periodic solution that occur by Hopf bifurcation. If the first Lyapunov coefficient is equal to zero, then a degenerated Hopf bifurcation takes place at α^*.

The construction of an approximation of the center manifold and the computation of the first Lyapunov coefficient (and thus of the normal form of the reduced equation) at a Hopf bifurcation point constitute the object of another paper of ours, [4].

b) The case $r|\delta+B|=1$, corresponds to the case $a_1=1$ from the paper of Hayes, [2]. In this case there always are eigenvalues with either positive or zero real part. The case of eigenvalues with zero real part (and all other with negative real part) corresponds to the case $a_2=-1$ of [2]. By using

the relations between a_1, a_2 and p, q, (these are $a_1 = -pr$, $a_2 = qr$) we find $pr = -1$, $qr = -1$, hence $\delta + B = kB = -1/r$. The first equality implies, as above, $\frac{\beta_0}{\delta}(k-1) = 1$ and it can not be satisfied in the zone of the parameters that we consider here. It follows that when $r|\delta + B| = 1$, the solution x_2 is unstable.

Remark. Assume that we vary r and keep all the other parameters fixed. The conditions (15) or (16) for r are not as simple as they seem, because B is itself a function of r (being a function of k). Let us consider the function

$$g(r) = T^{-1}(-(\delta + B(r))r) - \arccos\left(\frac{\delta + B(r)}{k(r)B(r)}\right). \tag{23}$$

If for a certain r^* we have $g(r^*) = 0$ (that is the condition for the change of stability), in order to find whether a value r_1 in a neighborhood of r^* is in the stability zone or not, we have to know the sign of $g(r_1)$, hence we have to study the monotony properties of function g in a neighborhood of r^*.

I.B. Since here $B < 0$, $B + \delta > 0$ we must have

$$\frac{\beta_0}{\delta}(k-1) > \frac{n}{n-1},$$
$$\frac{\beta_0}{\delta}(k-1)(n-k) < n. \tag{24}$$

The sufficient condition of local stability is condition (16), with B given by (17).

A point in the parameter space, satisfying

$$r = \frac{\arccos((\delta + B)/kB)}{\omega_0}$$

is a Hopf bifurcation point, if, with the notations from **I.A.a**, $\frac{d\mu}{d\alpha}(\alpha^*) \neq 0$ and the first Lyapunov coefficient of the reduced on the center manifold at α^* equation is different from zero. The stability of the solution in this case is given by the sign of the first Lyapunov coefficient.

Remark. If $B < 0$, $\delta + B = 0$, the solution is stable if and only if $-kBr < \pi/2$ while, for this case, the point $kBr = -\pi/2$ is a Hopf bifurcation point.

II. $B > 0$. If $n-1 < 0$, then $B > 0$. If $n-1 > 0$, then $B > 0$ is equivalent to

$$\frac{\beta_0}{\delta}(k-1) < \frac{n}{n-1}. \tag{25}$$

In this situation, all the eigenvalues have negative real part if and only if $kB < \delta + B$. This inequality is equivalent to

$$\frac{k-1}{\delta}B < 1 \Leftrightarrow \frac{1}{A}[n - (n-1)A] < 1 \Leftrightarrow A > 1 \Leftrightarrow \frac{\beta_0(k-1)}{\delta} > 1,$$

and this last inequality is already imposed (by the condition $x_2 > 0$. Hence in the case $B > 0$, x_2 is stable.

3 Comments on the stability results in [5], [6]

In order to compare our results with those of [5], [6], we define, for $n > 1$,

$$r_n := -\frac{1}{\gamma} \ln \left\{ \frac{1}{2} \left(\frac{\delta}{\beta_0} \frac{n}{n-1} + 1 \right) \right\}$$

and remark that $r_n > 0 \Leftrightarrow \frac{n}{n-1}\delta < \beta_0$. Also for $n > 1$, relation (25) implies that $B > 0 \Leftrightarrow r > r_n$. This last condition is trivially accomplished if $r_n \leq 0$ which is equivalent to $\frac{n}{n-1}\delta > \beta_0$.

With these remarks we get the following situations for the sign of B.

I. If $n < 1$ then $B > 0$.

II. If $n > 1$ and $\frac{n}{n-1}\delta > \beta_0$ then $B > 0$.

III. If $n > 1$ and $\frac{n}{n-1}\delta < \beta_0$ then

$$B > 0 \text{ for } r_n < r < r_{max},$$

$$B < 0 \text{ for } 0 < r < r_n.$$

This discussion allows us to follow the results of [5], [6] (the delay is there denoted by τ). Those results have the following weak points.

1. The results in [5] are presented for both equilibrium points simultaneously, and this leads to imprecisions. As example, the affirmation at point (1) in [5], pg. 238, is not true for $x_1 = 0$. Actually, the characteristic equation for this equilibrium point does not depend on n and thus for this point the condition $n \in [0, 1]$ is irrelevant. The condition of stability for this point does not depend on n. The ambiguity induced by using the plural "solutions" persists also at point (2) of [5], pg. 238, leading to misunderstandings since the conclusions there can not refer to x_1, as is seen from our Subsection 2.1.

2. In the case $B < 0$, the sign of $\delta + B$ is not considered in [5], pg. 238. To express the results of the analysis of the sign of $\delta + B$ in terms of r is a little more difficult since inequalities (19) and (24) contain second degree terms in k. However the cases $B + \delta > 0$ and $B + \delta < 0$ are different in conclusions and they can not be eluded.

3. The conclusions in [5], pg. 238, (2), b) seem to refer to the case $B < 0$, $\delta + B > 0$, but even for this case the result therein is not correct, since there the stability condition is

$$r < \frac{\arccos((B + \delta)/kB)}{\sqrt{(kB)^2 - (\delta + B)^2}}$$

instead of

$$r < \frac{\arccos((B + \delta)/kB)}{\omega_0}$$

with ω_0 defined in (13), as it is correct (condition (16)).

Since, in general, $\omega_0 \neq \sqrt{(kB)^2 - (\delta + B)^2}$ (equality holds, for $B < 0$, only when relation (20) holds), it is obvious that the domain of stability found in [5] is not correct (not even for the case $B < 0$, $\delta + B > 0$).

In [6], pgs. 316-317, the results are basically the same as in [5], excepting the fact that it seems that the discussion refers only to x_2 (but the plural "solutions" is used again). However, the observations from 2. and 3. above remain valid for [6] also.

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